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THEORY OF SMALL DEFORMATIONS OF PRESTRESSED THIN SHELLS

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Equations describing small deformations of prestressed shells are constructed on the basis of the three-dimensional theory of superposition of a small deformation on finite deformations. The equilibrium equations of prestressed shells are derived rigorously. The Kirchhoff hypotheses are taken for the additional displacements. An expression for the shell energy is obtained, after which the governing equations are formulated for shells of an arbitrary elastic material for any initial state of stress. The approach developed permits formulation also of the governing equations for shells of hypoelastic and elastoplastic materials. The governing relationships are made specific for the case of a small initial deformation. Boundary and conjugate conditions with absolutely rigid inclusions are derived from a variational principle. The case of additional external forces generated by the follower character of the hydrostatic load is specially considered. The results can be used in shell stability problems in particular.

1. Relationships of the theory of small deformations superposed on a finite deformation. The equilibrium conditions of a deformed material medium can be written as follows:

$$\iint\limits_{O_*} \mathbf{N}\cdot\mathrm{T}dO + \iiint\limits_{V_*} \rho\mathbf{b}d\tau = 0 \qquad (1.1)$$

$$\iint\limits_{O_*} \mathbf{R}\times(\mathbf{N}\cdot\mathrm{T})\,dO + \iiint\limits_{V_*} \rho\mathbf{R}\times\mathbf{b}d\tau = 0 \qquad (1.2)$$

Here T is the Cauchy stress tensor, $\mathbf{R}$ is the radius-vector of a point of the deformed body, V_* is an arbitrary volume isolated in the body, O_* is its bounding surface, $\mathbf{N}$ is the vector of the unit normal to the surface O_*, ρ is the density of the material in the deformed state, and $\mathbf{b}$ is the body force vector.

Let us consider some equilibrium state called the initial state and given by the vector $\mathbf{R}^\circ$, and an infinitely close equilibrium state given by the vector $\mathbf{R} = \mathbf{R}^\circ + \eta\mathbf{w}$, where η is a small parameter. Differentiating (1.1) and (1.2) with respect to the parameter η and setting $\eta = 0$, we obtain

$$\iint\limits_{O_*} \mathbf{N}^\circ\cdot\Theta dO + \iiint\limits_{V_*} \rho^\circ\mathbf{k}\,d\tau = 0 \qquad (1.3)$$

$$\iint\limits_{O_*} (\mathbf{N}^\circ\cdot\mathrm{T}^\circ\times\mathbf{w} + \mathbf{N}^\circ\cdot\Theta\times\mathbf{R}^\circ)\,dO + \qquad (1.4)$$

$$+\iiint\limits_{V_*} \rho^\circ(\mathbf{k}\times\mathbf{R}^\circ + \mathbf{b}^\circ\times\mathbf{w})\,d\tau = 0$$

$$\Theta = \mathrm{T} + \mathrm{T}^{\circ}\nabla\cdot\mathbf{w} - \nabla\mathbf{w}^T\cdot\mathrm{T}^{\circ}, \quad \mathrm{T}^{\cdot} = \left[\frac{d}{d\eta}\mathrm{T}\,(\mathbf{R}^{\circ} + \eta\mathbf{w})\right]_{\eta=0} \qquad (1.5)$$
$$\mathbf{b} = \mathbf{b}^{\circ} + \eta\mathbf{k}, \quad \mathbf{k} = \mathbf{b}^{\cdot}$$

Here Θ is the tensor introduced in [1, 2], ∇ is the nabla operator in the metric of the initial deformed state, $\mathbf{k}$ is the additional body force. From the arbitrariness of V_* in (1.3) and (1.4), and the equilibrium equation of the initial state

$$\nabla\cdot\mathrm{T}^{\circ} + \rho^{\circ}\mathbf{b}^{\circ} = 0 \qquad (1.6)$$

the equations

$$\nabla\cdot\Theta + \rho^{\circ}\mathbf{k} = 0, \quad \Theta - \Theta^{\mathrm{T}} = \mathrm{T}^{\circ}\cdot\nabla\mathbf{w} - \nabla\mathbf{w}^{\mathrm{T}}\cdot\mathrm{T}^{\circ} \qquad (1.7)$$

follow. The second relationship in (1.7) results directly from (1.5) also. Henceforth, the degree symbol referring to quantities in the initial deformed state is omitted. In the case of dead external forces, the potential energy accumulated in a prestressed elastic body with a small deformation is given by the expression [3]

$$\Pi = \iiint\limits_V W\,d\tau, \quad W = \frac{1}{2}\Theta\cdot\cdot\mathrm{L}^T, \quad \mathrm{L} = \nabla\mathbf{w} \qquad (1.8)$$

Decomposing the tensor L into symmetric and anti-symmetric components

$$\mathrm{L} = \varepsilon - \Omega, \quad \Omega = \mathrm{E}\times\boldsymbol{\omega} \qquad (1.9)$$

where ε is a linear strain tensor, $\boldsymbol{\omega}$ is a linear vector of rotation, and E is the unit tensor, we write the representation (1.5) of the tensor Θ as a linear function of the tensors ε and Ω as follows:

$$\Theta = \mathrm{P} + \mathrm{U}, \quad \mathrm{U} = {}^1/_2\,(\mathrm{T}\cdot\varepsilon - \varepsilon\cdot\mathrm{T}) - \mathrm{T}\cdot\Omega \qquad (1.10)$$
$$\mathrm{P} = \mathrm{P}^T = \mathrm{T}^{\cdot} + \mathrm{T}\nabla\cdot\mathbf{w} - \mathrm{L}^T\cdot\mathrm{T} - \mathrm{T}\cdot\mathrm{L} + {}^1/_2\,(\varepsilon\cdot\mathrm{T} + \mathrm{T}\cdot\varepsilon)$$

The symmetric tensor P which is the objective derivative [4] of the objective tensor T is independent of the rotation tensor Ω. The potential energy density becomes the following:

$$W = W^P + W^U, \quad W^P = {}^1/_2\mathrm{P}\,(\varepsilon)\cdot\cdot\varepsilon \qquad (1.11)$$
$$W^U = {}^1/_2\mathrm{U}\cdot\cdot\,\mathrm{L}^{\mathrm{T}} = \mathrm{tr}\,(\Omega\cdot\mathrm{T}\cdot\varepsilon - {}^1/_2\Omega\cdot\mathrm{T}\cdot\Omega) = {}^1/_8\mathrm{tr}\,(3\mathrm{L}^T\cdot\mathrm{T}\cdot\mathrm{L} - 2L^T\cdot\mathrm{T}\cdot\mathrm{L}^T - \mathrm{L}\cdot\mathrm{T}\cdot\mathrm{L}^T)$$

It is understood that the tensor P still depends on the initial deformation. The formulas

$$\Theta = \partial W/\partial\mathrm{L}, \quad \mathrm{U} = \partial W^U/\partial\mathrm{L}, \quad \mathrm{P} = \partial W^P/\partial\varepsilon \qquad (1.12)$$

are valid for an ideal elastic body. The first of the relationships (1.12) has been obtained in [3], the second is verified directly by using (1.11), and the third follows from the first and second.

The relationships (1.10) — (1.12) show that the specific energy of small displacements of a prestressed body consists of the pure strain energy $W^P(\varepsilon)$ and the energy W^U due to rotation of a volume element with small displacements. For an elastic body the coefficients of the quadratic form $W^P(\varepsilon)$ are determined completely by the law of the material state and the initial strains. The energy W^U is independent of the material properties and determined entirely by the initial stresses. The material properties can also be given directly as a linear dependence of the tensor P on the tensor ε with coefficients dependent on initial stresses (plastic flow theory), where this representation will be different in cases of active loading and unloading.

For an absolutely rigid body we have $\varepsilon = 0$, $\omega = \text{const}$. Hence, according to (1.11), the potential energy of small displacements of a solid is represented for dead external forces as

$$\Pi = -\frac{1}{2}\iiint_V \operatorname{tr}(\omega \times \mathrm{T} \times \omega)\, d\tau = \frac{1}{2}\omega \cdot \iiint_V (\mathrm{E}\operatorname{tr}\mathrm{T} - \mathrm{T})\, d\tau \cdot \omega \quad (1.13)$$

The stress tensor is not defined in an absolutely rigid body, however, by using the equilibrium equations (1.6), the integral in (1.13) can be expressed in terms of the external forces

$$\iiint_V (\mathrm{E}\operatorname{tr}\mathrm{T} - \mathrm{T})\, d\tau = \iiint_V (\rho\mathbf{b}\cdot\mathbf{RE} - \rho\mathbf{bR})\, d\tau + \quad (1.14)$$

$$\iint_O (\mathbf{F}\cdot\mathbf{RE} - \mathbf{FR})\, dO \equiv \Phi$$

Here $\mathbf{F}$ is the surface force vector. Since the system of forces $\mathbf{F}$ and $\mathbf{b}$ is statically equivalent to zero, the tensor Φ is symmetric. We have from (1.13) and (1.14)

$$\Pi = {}^1/_2\, \omega\cdot\Phi\cdot\omega \quad (1.15)$$

The condition of stationarity of the functional Π is equivalent to the neutral equilibrium conditions. Applied to (1.15), we obtain $\delta\Pi = \omega\cdot\Phi\cdot\delta\omega$. From the condition $\delta\Pi = 0$ we arrive at an equation to determine the equilibrium axis [5], i.e. that vector ω for rotation around which the solid remains in equilibrium: $\Phi\cdot\omega = 0$.

2. Equilibrium equations of a prestressed shell. Let a surface O referred to the Gaussian coordinates q^1, q^2 be the middle surface of the shell in the initial deformed state. The vectors of the fundamental and reciprocal bases on the surface are given by the formulas

$$\mathbf{R}_\alpha = \partial\mathbf{R}_0 / \partial q^\alpha, \quad \mathbf{R}^\beta\cdot\mathbf{R}_\alpha = \delta_\alpha{}^\beta, \quad \mathbf{R}^\beta\cdot\mathbf{N} = 0 \quad (\alpha, \beta = 1, 2)$$

Here $\mathbf{R}_0$ is the radius-vector of a point of the surface, $\mathbf{N}$ is the vector of the unit normal to the surface. We introduce the first and second fundamental tensors of the surface by the relationships

$$\mathrm{G} = G_{\alpha\beta}\mathbf{R}^\alpha\mathbf{R}^\beta = \mathrm{E} - \mathbf{NN}, \quad \mathbf{B} = B_{\alpha\beta}\mathbf{R}^\alpha\mathbf{R}^\beta = -\nabla_2\mathbf{N} = \mathbf{B}^T \quad (2.1)$$

$$G_{\alpha\beta} = \mathbf{R}_\alpha\cdot\mathbf{R}_\beta, \quad B_{\alpha\beta} = -\mathbf{R}_\beta\cdot\partial N / \partial q^\alpha, \quad \nabla_2 = \mathbf{R}^\alpha\partial / \partial q^\alpha$$

Here ∇_2 is the nabla-operator on the surface. Let us say that the Euclidean tensor of the second rank Y belongs to the surface if it satisfies the relationship $\mathrm{N}\cdot\mathrm{Y} = \mathrm{Y}\cdot\mathrm{N} = 0$. Evidently the tensors G and B belong to the surface.

The shell is a domain bounded by two surfaces on different sides of the middle surface O at an identical distance away $h / 2$, and by a ruled surface Σ formed by motion of the normal to the middle surface along its boundary outline. The position of points of the shell is given by the radius-vector

$$\mathbf{R} = \mathbf{R}_0 + z\mathbf{N}, \quad -h / 2 \leqslant z \leqslant h / 2$$

The shell thickness can be variable.

The validity of the following formulas which express the geometric characteristics of the shell in terms of the geometric characteristics of the middle surface can be shown:

$$\mathbf{m}d\Sigma = A\,(\mathrm{G} - z\mathrm{B})^{-1}\cdot\mathbf{m}_0 dS dz, \quad A = \det(\mathrm{G} - z\mathrm{B}) \quad (2.2)$$

$$d\tau = A dO dz \tag{2.3}$$

$$\nabla\varphi(q^1, q^2, z) = (\mathbf{G} - z\mathbf{B})^{-1}\cdot\nabla_2\varphi + \mathbf{N}\partial\varphi/\partial z \tag{2.4}$$

$$(\mathbf{N}dO)_\pm = A(\pm h/2)[\pm\mathbf{N} \mp 1/2(\mathbf{G} \mp h/2\mathbf{B})^{-1}\cdot\nabla_2 h]dO \tag{2.5}$$

Here $\mathbf{m}$ is the unit normal to the surface Σ, $\mathbf{m}_0$ is the external normal to the boundary outline of the middle surface ($\mathbf{m}_0\cdot\mathbf{N} = 0$), dO is an element of the middle surface, dS is an element of arc of the outline, φ is an arbitrary function of the coordinates (not necessarily a scalar), $(\mathbf{N}dO)_\pm$ is a directed area element of the surfaces $z = \pm h/2$. The tensor $\mathbf{Y}^{-1}$, inverse to the tensor $\mathbf{Y}$ belonging to the surface, is understood to be a tensor which also belongs to the surface and satisfies the relationship

$$\mathbf{Y}^{-1}\cdot\mathbf{Y} = \mathbf{Y}\cdot\mathbf{Y}^{-1} = \mathbf{G}$$

In addition to the external forces due to the initial equilibrium state, let small additional external surface forces distributed over the surfaces $z = \pm h/2$ and additional forces distributed over the shell volume also act on the shell. Within each section of the shell resting on the element dO of the middle surface, we replace this system of additional forces by its statically equivalent system of forces and moments concentrated on the middle surface with the intensities $\mathbf{f}$ and $\boldsymbol{\mu}\times\mathbf{N}$ ($\boldsymbol{\mu}\cdot\mathbf{N} = 0$) per unit area, respectively. We apply the equilibrium conditions (1.3),(1.4) to the shell by taking a section of the shell bounded by the surfaces $z = \pm h/2$ and the surface Σ_* intersecting the middle surface along an arbitrary outline Γ_* as V_*. We obtain

$$\iint_{\Sigma_*}\mathbf{m}\cdot\Theta d\Sigma + \iint_{O_*}\mathbf{f}\,dO = 0 \tag{2.6}$$

$$\iint_{\Sigma_*}\mathbf{m}\cdot\Theta\times\mathbf{R}d\Sigma + \iint_{O_*}(\mathbf{f}\times\mathbf{R} + \boldsymbol{\mu}\times\mathbf{N})dO + \iint_{\Sigma_*}\mathbf{m}\cdot\mathbf{T}\times\mathbf{w}d\Sigma + \tag{2.7}$$

$$\iint_{O_*}\mathbf{N}_+\cdot\mathbf{T}\times\mathbf{w}dO_+ + \iint_{O_{*-}}\mathbf{N}_-\cdot\mathbf{T}\times\mathbf{w}dO_- +$$

$$\iint_{O_*}\Big[\int_{-h/2}^{h/2}A\rho\mathbf{b}\times\mathbf{w}dz\Big]dO = 0$$

Applying (2.2)—(2.5), the theorem about the divergence on the surface, taking account of the equilibrium differential equations for the initial stresses (1.6), and using the arbitrariness of the surface O_*, we obtain from (2.6) and (2.7)

$$\nabla_2\cdot\mathbf{K}' + \mathbf{N}\nabla_2\cdot\mathbf{V} - \mathbf{V}\cdot\mathbf{B} + \mathbf{f} = 0 \tag{2.8}$$

$$[\mathbf{K}' + (\mathbf{B}\cdot\mathbf{M}')^T]\cdot\cdot\in_2\mathbf{N} + (\nabla_2\cdot\mathbf{M}')\times\mathbf{N} - \mathbf{V}\times\mathbf{N} + \boldsymbol{\mu}\times\mathbf{N} = \mathbf{l} \equiv \tag{2.9}$$

$$-\int_{-h/2}^{h/2}A[\mathbf{R}^\alpha\cdot(\mathbf{G} - z\mathbf{B})^{-1}\cdot\mathbf{T}\times\partial\mathbf{w}/\partial q^\alpha + \mathbf{N}\cdot\mathbf{T}\times\partial\mathbf{w}/\partial z]dz$$

$$\in_2 = -\in_2^T = \in\cdot\mathbf{N} = -\mathbf{G}\times\mathbf{N}, \quad \in = -\mathbf{E}\times\mathbf{E}$$

$$\begin{bmatrix}\mathbf{K}'\\ \mathbf{M}'\end{bmatrix} = \int_{-h/2}^{h/2}A(\mathbf{G} - z\mathbf{B})^{-1}\cdot\Theta\cdot\mathbf{G}\begin{bmatrix}1\\ z\end{bmatrix}dz \tag{2.10}$$

$$\mathbf{V} = \int_{-h/2}^{h/2} A(\mathrm{G} - z\mathrm{B})^{-1}\cdot\Theta\cdot\mathrm{N}dz$$

Here $\in_2$ is the discriminant surface tensor, $\in$ is the three-dimensional discriminant tensor (the Levi-Civitã tensor). Introducing the notation for a vector in the right side of (2. 9)

$$-\mathrm{I} = \gamma\mathrm{N} + \lambda \times \mathrm{N} \quad (\lambda\cdot\mathrm{N} = 0) \tag{2.11}$$

we see that (2. 9) decomposes into the two equations

$$[\mathrm{K}' + (\mathrm{B}\cdot\mathrm{M}')^T] \cdot\cdot \in_2 + \gamma = 0 \tag{2.12}$$

$$(\nabla_2\cdot\mathrm{M}')\cdot\mathrm{G} - \mathrm{V} + \mu + \lambda = 0 \tag{2.13}$$

Introducing the symmetric tensor K into the considerations, we have from (2. 12)

$$\mathrm{K}' = \mathrm{K} - (\mathrm{B}\cdot\mathrm{M}')^T + {}^1/_2\gamma \in_2 \tag{2.14}$$

$$\mathrm{K} = {}^1/_2 [\mathrm{K}' + (\mathrm{B}\cdot\mathrm{M}')^T] + {}^1/_2 [\mathrm{K}'^T + \mathrm{B}\cdot\mathrm{M}']$$

Eliminating the vector V from (2. 13) and (2. 8) and using (2. 14), we arrive at a vector equation. On the basis of the Ricci identity and the properties of the Riemann-Christoffel tensor of a surface [6], it can be proved that only the symmetric part of the tensor M′ enters into this equation. We then finally obtain the following equilibrium equation for a prestressed shell

$$\begin{aligned} &\nabla_2\cdot\mathrm{K} - 2\mathrm{B}\cdot(\nabla_2\cdot\mathrm{M}) - (\mathrm{M}\cdot\nabla_2)\cdot\mathrm{B} + \mathrm{N}\nabla_2\cdot [\mathrm{G}\cdot (\nabla_2\cdot\mathrm{M})] - \\ &\quad {}^1/_2 \in_2\cdot\nabla_2\gamma - \mathrm{B}\cdot (\mu + \lambda) + \mathrm{N}\nabla_2\cdot(\mu + \lambda) + \mathrm{f} = 0 \\ &\mathrm{M} = {}^1/_2 (\mathrm{M}' + \mathrm{M}'^T) \end{aligned} \tag{2.15}$$

These equations are exact consequences of the necessary conditions for equilibrium (1. 3) and (1. 4).

The following relationship

$$\int_{-h/2}^{h/2} (\mathrm{N}\cdot\Theta - \Theta\cdot\mathrm{N})\,dz = -\mathrm{N} \times (\lambda \times \mathrm{N}) = -\lambda \tag{2.16}$$

can be obtained from (1. 7), (2. 4) and 2. 11).

For small additional displacements of points of the shell we use the Kirchhoff kinematic hypotheses

$$\mathrm{w} = \mathrm{w}_0 - z\vartheta, \quad \vartheta = -\mathrm{N}^{\cdot} = \mathrm{N}\cdot\nabla_2\mathrm{w}_0{}^T = \nabla_2 w + \mathrm{B}\cdot\mathrm{u} \tag{2.17}$$

$$\mathrm{u} = \mathrm{w}_0\cdot\mathrm{G}, \quad w = \mathrm{w}_0\cdot\mathrm{N}$$

Here w_0 is the middle surface displacement vector. According to (2. 4), after some transformations we obtain for the tensor L of the shell

$$\mathrm{L} = (\mathrm{G} - z\mathrm{B})^{-1}\cdot[\varepsilon_0 - z(\varkappa - \mathrm{B}\cdot\varepsilon_0)] + \chi\in_2 + \vartheta\mathrm{N} - \mathrm{N}\vartheta \tag{2.18}$$

$$\begin{aligned} &\varepsilon_0 = {}^1/_2 [(\nabla_2\mathrm{u})\cdot\mathrm{G} + \mathrm{G}\cdot(\nabla_2\mathrm{u})^T] - \mathrm{B}w \\ &\varkappa = (\nabla_2\vartheta)\cdot\mathrm{G} + \mathrm{B}\cdot\nabla_2\mathrm{w}_0^T, \quad \chi = {}^1/_2\,\mathrm{N}\cdot(\nabla_2 \times \mathrm{u}) \end{aligned} \tag{2.19}$$

It can be confirmed that the tensors ε_0 and $\varkappa$ belong to the surface O and are symmetric. They define infinitesimal changes in the metric and curvature of the surface, respectively. The quantity χ characterizes the infinitesimal rotation of a surface element around a normal. The vector $\boldsymbol{\vartheta}$ belonging to the surface characterizes the angle of rotation of the normal for small surface displacements.

It must be kept in mind that the Kirchhoff hypotheses (2. 17) should not be considered as the requirement of no transverse strain $\varepsilon_{33} = \mathbf{N}\cdot\varepsilon\cdot\mathbf{N}$. This strain should be determined from the condition

$$\theta_{33} = \mathbf{N}\cdot\Theta\cdot\mathbf{N} = 0 \tag{2.20}$$

Taking account of this circumstance, and using the representation (2. 18), we obtain from (2. 11) after some calculations

$$\gamma = \mathrm{tr}\,[\mathrm{S}\cdot\in_2\cdot\varepsilon_0 - \mathrm{D}\cdot\in_2\cdot(\varkappa - \varepsilon_0\cdot\mathrm{B}) + \mathrm{S}\chi - \mathrm{D}\cdot\mathrm{B}\chi] + \boldsymbol{\vartheta}\cdot\in_2\cdot\mathbf{Q} - \boldsymbol{\vartheta}\cdot\in_2\cdot\mathrm{B}\cdot\mathbf{q} \tag{2.21}$$

$$\boldsymbol{\lambda} = -\mathbf{Q}\cdot\varepsilon_0 + \mathbf{q}\cdot(\varkappa - \mathrm{B}\cdot\varepsilon_0) - \chi\mathbf{Q}\cdot\in_2 + \chi\mathbf{q}\cdot\mathrm{B}\cdot\in_2 + \boldsymbol{\vartheta}\cdot\mathrm{S} - \boldsymbol{\vartheta}\cdot\mathrm{B}\cdot\mathrm{D} + \psi\boldsymbol{\vartheta} + \boldsymbol{\xi} \tag{2.22}$$

$$\begin{bmatrix}\mathrm{S}\\ \mathrm{D}\end{bmatrix} = \int_{-h/2}^{h/2} A\,(\mathrm{G} - z\mathrm{B})^{-1}\cdot\mathrm{T}\cdot\mathrm{G}\begin{bmatrix}1\\ z\end{bmatrix}dz, \quad \psi = \int_{-h/2}^{h/2} A\mathbf{N}\cdot\mathrm{T}\cdot\mathbf{N}dz \tag{2.23}$$

$$\boldsymbol{\xi} = \int_{-h/2}^{h/2} A\mathbf{N}\cdot\mathrm{T}\cdot\mathrm{G}\varepsilon_{33}dz, \quad \begin{bmatrix}\mathbf{Q}\\ \mathbf{q}\end{bmatrix} = \int_{-h/2}^{h/2} A\,(\mathrm{G} - z\mathrm{B})^{-1}\cdot\mathrm{T}\cdot\mathbf{N}\begin{bmatrix}1\\ z\end{bmatrix}dz$$

Here S and D are, respectively, the initial force and moments tensors in the shell, $\mathbf{Q}$ is the transverse force vector in the initial stress state. The relationships (2. 21) and (2. 22) are exact corollaries of the Kirchhoff hypotheses.

3. Specific strain energy and governing equations of a prestressed shell. The shell potential strain energy per unit middle surface area is given according to (1. 8), (2. 3) and (1. 11) by the expression

$$a = \int_{-h/2}^{h/2} A\,(W^P + W^U)\,dz = a^P + a^U \tag{3.1}$$

We obtain from (1. 12), (2. 10), (2. 18), (2. 20), (2. 12) and (2. 16) for the energy variation

$$\delta a = \mathrm{K}\cdot\cdot\,\delta\varepsilon_0 - \mathrm{M}\cdot\cdot\,\delta\varkappa + \gamma\delta\chi + \boldsymbol{\lambda}\cdot\delta\boldsymbol{\vartheta} \tag{3.2}$$

The governing equations of a prestressed shell follow from (3. 2)

$$\mathrm{K} = \partial a\,(\varepsilon_0, \varkappa, \chi, \boldsymbol{\vartheta})\,/\,\partial\varepsilon_0, \quad \mathrm{M} = -\partial a\,(\varepsilon_0, \varkappa, \chi, \boldsymbol{\vartheta})\,/\,\partial\varkappa \tag{3.3}$$

The energy a^U is calculated by using (1. 11) and (2. 18) and is for any material, to the accuracy of terms containing the integral σ_1:

$$a^U = \mathrm{S}\cdot\cdot\,(\chi\in_2\cdot\varepsilon_0 + {}^1/_2\chi^2\mathrm{G} + {}^1/_2\boldsymbol{\vartheta}\boldsymbol{\vartheta}) + \mathrm{D}\cdot\cdot\,({}^3/_4\mathrm{B}\cdot\varepsilon_0^2 + {}^1/_4\varepsilon_0^2\cdot\mathrm{B} - \varepsilon_0\cdot\mathrm{B}\cdot\varepsilon_0 - \chi\in_2\cdot\varkappa + \chi\in_2\cdot\varepsilon_0\cdot\mathrm{B} - {}^1/_2\chi^2\mathrm{B} - {}^1/_2\boldsymbol{\vartheta}\boldsymbol{\vartheta}\cdot\mathrm{B}) + \mathbf{Q}\cdot(-\varepsilon_0\cdot\boldsymbol{\vartheta} - \chi\in_2\cdot\boldsymbol{\vartheta}) + \mathbf{q}\cdot(\varkappa\cdot\boldsymbol{\vartheta} - \mathrm{B}\cdot\varepsilon_0\cdot\boldsymbol{\vartheta} + \chi\mathrm{B}\cdot\in_2\cdot\boldsymbol{\vartheta}) + {}^1/_2\psi\boldsymbol{\vartheta}\cdot\boldsymbol{\vartheta} + \boldsymbol{\xi}\cdot\boldsymbol{\vartheta} + \sigma\cdot\cdot\,({}^3/_2\varkappa\cdot\mathrm{B}\cdot\varepsilon_0 - {}^1/_2\varepsilon_0\cdot\varkappa\cdot\mathrm{B} - \mathrm{B}\cdot\varepsilon_0\cdot\varkappa + {}^3/_4\mathrm{B}^2\cdot\varepsilon_0^2 + {}^3/_4\mathrm{B}\cdot\varepsilon_0^2\cdot\mathrm{B} - {}^3/_2\varepsilon_0\cdot\mathrm{B}^2\cdot\varepsilon_0) \tag{3.4}$$

$$\sigma = \int_{-h/2}^{h/2} \mathrm{G}\cdot\mathrm{T}\cdot\mathrm{G}z^2dz, \quad \sigma_1 = \int_{-h/2}^{h/2} \mathrm{G}\cdot\mathrm{T}\cdot\mathrm{G}z^3dz$$

It should be noted that the terms discarded in (3.4) are independent of χ, ϑ exactly as the terms containing the tensor σ .

By using the relationships (1.10),(1.11),(2.18) and (2.20) it can be confirmed from (3.4) that the formulas

$$\gamma = \partial a / \partial \chi, \qquad \lambda = \partial a / \partial \vartheta$$

follow in complete conformity with (3.2),(2.21) and (2.22).

The terms dependent on ψ, q, σ can be neglected in (3.4) and therefore in (2.21) and (2.22) also in the majority of practical cases.

The shell energy a^P can be evaluated if the law of the material state and the initial strain are known; for an elastic material $P(\varepsilon) = C \cdot\cdot\, \varepsilon$, where the tensor of the fourth rank C depends on the initial strain.

In the case of an isotropic material, the tensor C is a function of the Almansi strain tensor [1] of the initial state and can be represented as an expansion in powers of the Almansi tensor, where the zero term of this expansion is (λ, μ are Lamé constants)

$$\overset{\circ}{C}_{mnlk} = \lambda\delta_{mn}\delta_{kl} + \mu\,(\delta_{mk}\delta_{nl} + \delta_{ml}\delta_{nk})$$

If the principal relative elongations are small in the initial state (the displacements and rotations can be finite), then only the zero term can be retained in the expansion mentioned. Successive evaluation of the remaining terms in the expansion is impossible without knowing the second and higher order elastic constants of the material.

It can be assumed that for a small initial strain the tensor P is related to the tensor ε by Hooke's law

$$P(\varepsilon) = \lambda \operatorname{tr}\varepsilon\, E + 2\mu\varepsilon \tag{3.5}$$

Let us take the following type of distribution of the initial tangential stresses over the thickness

$$N\cdot T\cdot G \approx \frac{3}{2h}\left[1 - \left(\frac{2z}{h}\right)^2\right] Q$$

Then after calculations using the formulas (3.1),(3.5),(1.11),(2.18) and (2.20), we have to a sufficient degree of accuracy [7]

$$\begin{aligned} a^P = {} & \frac{Eh}{2(1-\nu^2)}\,[\operatorname{tr}^2 \varepsilon_0 - 2(1-\nu)\det \varepsilon_0] + \\ & \frac{Eh^3}{24(1-\nu^2)}\,[\operatorname{tr}^2 \varkappa - 2(1-\nu)\det \varkappa] + \frac{3(1+\nu)(1-2\nu)}{5Eh(1-\nu)}\,(\vartheta\cdot Q)^2 \\ \xi = {} & -\frac{\nu}{1-\nu}\,Q \operatorname{tr}\varepsilon_0 - \frac{6(1+\nu)(1-2\nu)}{5Eh(1-\nu)}\,QQ\cdot\vartheta \end{aligned} \tag{3.6}$$

Here E is the Young's modulus and ν is the Poisson's ratio. Neglecting terms containing ψ, q, σ we obtain representations of the tensors K and M from (3.3),(3.4) and (3.6):

$$\begin{aligned} K = {} & \frac{Eh}{(1-\nu^2)}\,[(1-\nu)\,\varepsilon_0 + \nu G \operatorname{tr}\varepsilon_0] + \\ & \frac{1}{2}\chi\,(S\cdot\in_2 - \in_2\cdot S^T + B\cdot D\cdot\in_2 - \in_2\cdot D^T\cdot B) - \\ & \frac{1}{2}(D + D^T)\cdot\varepsilon_0\cdot B - \frac{1}{2}\,B\cdot\varepsilon_0\cdot(D + D^T) + \\ & \frac{3}{8}(D\cdot B + B\cdot D^T)\cdot\varepsilon_0 + \frac{3}{8}\,\varepsilon_0\cdot(D\cdot B + B\cdot D^T) + \\ & \frac{1}{8}(D^T\cdot B + B\cdot D)\cdot\varepsilon_0 + \frac{1}{8}\varepsilon_0\cdot(D^T\cdot B + B\cdot D) - \end{aligned} \tag{3.7}$$

$$\frac{1}{2}(\vartheta\mathrm{Q}+\mathrm{Q}\vartheta)-\frac{\nu}{1-\nu}\,\mathrm{Q}\cdot\vartheta\mathrm{G}$$

$$\mathrm{M}=-\frac{Eh^3}{12(1-\nu^2)}\left[(1-\nu)\varkappa+\nu\mathrm{G}\,\mathrm{tr}\,\varkappa\right]-\frac{1}{2}\chi\left(\in_2\cdot\mathrm{D}^T-\mathrm{D}\cdot\in_2\right)$$

The equilibrium equations (2. 15), together with the relationships (2. 21), (2. 22) and the governing equations (3. 7), form a complete system of equations for a prestressed shell in the case of a small initial strain. It can be seen that this system of equations reduces to the classical Bryan equation [8] in application to the problem of bending of a slab prestressed in its plane.

4. Boundary conditions. The governing equations and force boundary conditions of the problem of deformation of a prestressed body can be obtained from the variational principle [3]

$$\delta\iiint\limits_V W\,d\tau-\iiint\limits_V \rho\mathbf{k}\cdot\delta\mathbf{w}\,d\tau-\iint\limits_O \mathbf{f}\cdot\delta\mathbf{w}\,dO=0 \tag{4.1}$$

Let the middle surface O be bounded by a smooth outline Γ on which the additional external force with intensity $\mathbf{l}$ per unit length and the moment with intensity $\mathbf{d}\times\mathbf{N}$ $(\mathbf{d}\cdot\mathbf{N}=0)$ are given, as well as by an outline Γ' along which an absolutely rigid core adjoins the shell middle surface. The outline Γ' separates the surface of a rigid inclusion into two parts. To simplify the formulas, we assume that the additional external forces are applied just to one of these parts designated O'.

From (4. 1), (3. 2) and (1. 15) we obtain

$$\iint\limits_O(\mathrm{K}\cdot\cdot\,\delta\varepsilon_0-\mathrm{M}\cdot\cdot\,\delta\varkappa+\gamma\delta\chi+\boldsymbol{\lambda}\cdot\delta\boldsymbol{\vartheta}-\mathbf{f}\cdot\delta\mathbf{w}_0+\boldsymbol{\mu}\cdot\delta\boldsymbol{\vartheta})\,dO- \tag{4.2}$$

$$\iint\limits_{O'}\mathbf{f}\cdot\delta\mathbf{w}\,dO+\boldsymbol{\omega}\cdot\boldsymbol{\Phi}\cdot\delta\boldsymbol{\omega}-\int\limits_\Gamma(\mathbf{l}\cdot\delta\dot{\mathbf{w}}_0-\mathbf{d}\cdot\delta\boldsymbol{\vartheta})\,dS=0$$

Here $\boldsymbol{\omega}$ is the vector of infinitesimal rotations of the rigid core.

The tensor $\boldsymbol{\Phi}$ specifying the core potential energy has two components $\boldsymbol{\Phi}=\boldsymbol{\Phi}_1+\boldsymbol{\Phi}_2$. The tensor $\boldsymbol{\Phi}_1$ is generated by reactive forces acting on the core in the initial state of stress and exerted by the shell and $\boldsymbol{\Phi}_2$ is calculated by means of (1. 14) in terms of the external forces of the initial state applied to the core. On the basis of (1. 14), (2. 2) and (2. 23) we have

$$\boldsymbol{\Phi}_1=-\int\limits_{\Gamma'}[\mathbf{m}_0\cdot(\mathrm{S}\cdot\mathbf{R}_0+\mathrm{Q}\mathbf{N}\cdot\mathbf{R}_0+\mathbf{q})\,\mathrm{E}-(\mathbf{m}_0\cdot\mathrm{S})\,\mathbf{R}_0- \tag{4.3}$$

$$(\mathbf{m}_0\cdot\mathrm{D})\,\mathbf{N}-(\mathbf{m}_0\cdot\mathbf{Q})\,\mathbf{N}\mathbf{R}_0-(\mathbf{m}_0\cdot\mathbf{q})\,\mathbf{N}\mathbf{N}]\,dS$$

Here $\mathbf{m}_0$ is the normal to the outline Γ', external with respect to the domain O occupied by the shell. Using (2. 19) and integrating by parts, we obtain instead of (4. 2)

$$-\iint\limits_O\Big\{\nabla_2\cdot\mathrm{K}-\nabla_2\cdot(\mathrm{M}\cdot\mathbf{B})-(\nabla_2\cdot\mathrm{M})\cdot\mathbf{B}+\frac{1}{2}\nabla_2\cdot(\gamma\in_2)- \tag{4.4}$$

$$(\boldsymbol{\lambda}+\boldsymbol{\mu})\cdot\mathbf{B}+\mathbf{N}\nabla_2\cdot[\mathrm{G}\cdot(\nabla_2\cdot\mathrm{M})]+\mathbf{N}\nabla_2\cdot(\boldsymbol{\lambda}+\boldsymbol{\mu})+\mathbf{f}\Big\}\cdot\delta\mathbf{w}_0\,dO+$$

$$\int\limits_\Gamma\Big[\Big\{\mathbf{m}_0\cdot(\nabla_2\cdot\mathrm{M})+\mathbf{m}_0\cdot(\boldsymbol{\lambda}+\boldsymbol{\mu})-\mathbf{l}\cdot\mathbf{N}+\frac{\partial M_{mt}}{\partial S}-\frac{\partial d_t}{\partial S}\Big]\delta w+$$

$$\Big[\mathbf{m}_0\cdot(\mathrm{K}-2\mathrm{M}\cdot\mathbf{B})+\mathbf{d}\cdot\mathbf{B}-\mathbf{l}\cdot\mathrm{G}+\frac{1}{2}\gamma\mathbf{m}_0\cdot\in_2\Big]\cdot\delta\mathbf{u}-$$

$$(M_{mm}-d_m)\,\delta\left(\frac{\partial w}{\partial m}\right)\Big\}\,dS+\boldsymbol{\omega}\cdot\boldsymbol{\Phi}\cdot\delta\boldsymbol{\omega}+$$

$$\int_{\Gamma'} \left\{ \mathbf{m}_0 \cdot \left(\mathrm{K} - 2\mathrm{M}\cdot\mathbf{B} + \frac{1}{2}\,\gamma \cdot \in_2 \right) \cdot \delta\mathbf{u} + \left[\mathbf{m}_0 \cdot (\nabla_2 \cdot \mathrm{M}) + \right.\right.$$
$$\left.\left. \mathbf{m}_0 \cdot (\boldsymbol{\lambda} + \boldsymbol{\mu}) + \frac{\partial M_{mt}}{\partial S} \right] \delta w - M_{mm} \delta \left(\frac{\partial w}{\partial m} \right) \right\} dS - \iint_{O'} \mathbf{f} \cdot \delta\mathbf{w}\, dO = 0$$

The kinematic conditions of connection between the shell and the core are written as follows:

$$\mathbf{w}_0 = \mathbf{v} - \mathbf{R}_0 \times \boldsymbol{\omega}, \quad \delta\mathbf{w}_0 = \delta\mathbf{v} - \mathbf{R}_0 \times \delta\boldsymbol{\omega} \tag{4.5}$$
$$\mathbf{m}_0 \cdot \boldsymbol{\vartheta} = -\mathbf{t}\cdot\boldsymbol{\omega}, \quad \delta(\partial w / \partial m) = -\mathbf{m}_0 \cdot \mathbf{B} \cdot \delta\mathbf{v} +$$
$$(-\mathbf{t} + m_0 \cdot \mathbf{B} \times \mathbf{R}_0) \cdot \delta\boldsymbol{\omega}, \quad \mathbf{t} = -\mathbf{m}_0 \times \mathbf{N}$$

Here $\mathbf{v}$ is the vector of translational displacements of the core. By equating the coefficients in the variations $\delta\mathbf{w}_0$ on O, $\delta\mathbf{w}_0$ and $\delta\,(\partial w / \partial m)$ on Γ to zero, we arrive at the equilibrium equations (2.15) and the force boundary conditions on Γ. Furthermore, substituting (4.5) into (4.4) and equating the coefficients of the variations $\delta\mathbf{v}$ and $\delta\boldsymbol{\omega}$ to zero, we obtain the conditions for connection to a rigid core

$$\int_{\Gamma'} \left\{ \mathbf{m}_0 \cdot \left(\mathrm{K} - 2\mathrm{M}\cdot\mathbf{B} + \frac{1}{2}\gamma \in_2 + \mathbf{B} M_{mm} \right) + \right. \tag{4.6}$$
$$\left. \mathbf{m}_0 \cdot (\nabla_2 \cdot \mathrm{M} + \boldsymbol{\lambda} + \boldsymbol{\mu})\, \mathbf{N} + \frac{\partial M_{mt}}{\partial S}\, \mathbf{N} \right\} dS - \iint_{O'} \mathbf{f}\, dO = 0$$
$$\int_{\Gamma'} \left\{ \mathbf{m}_0 \cdot \left(\mathrm{K} - 2\mathrm{M}\cdot\mathbf{B} + \frac{1}{2}\gamma \in_2 + \mathbf{B} M_{mm} \right) \times \mathbf{R}_0 + \right.$$
$$\left. \mathbf{m}_0 \cdot \left(\nabla_2 \cdot \mathrm{M} + \boldsymbol{\lambda} + \boldsymbol{\mu} + \frac{\partial M_{mt}}{\partial S} \right) \mathbf{N} \times \mathbf{R}_0 - \mathbf{t} M_{mm} \right\} dS +$$
$$\iint_{O'} \mathbf{R} \times \mathbf{f}\, dO - \boldsymbol{\omega} \cdot \boldsymbol{\Phi} = 0$$

The additional load $\mathbf{f}$ should not be considered given necessarily, it can depend on the displacement vector $\mathbf{w}$. If a hydrostatic pressure of intensity p acts on the body surface in the initial strain state, then upon going over to a new state, an additional surface load originates [1]

$$\mathbf{f} = -p(\nabla \cdot \mathbf{w}\mathrm{E} - \nabla\mathbf{w}) \cdot \mathbf{N} = p\mathbf{N} \cdot (\nabla_2 \mathbf{w}_0^T - \mathrm{E}\nabla_2 \cdot \mathbf{w}_0) = p\,(\boldsymbol{\vartheta} - \mathbf{N}\,\mathrm{tr}\,\varepsilon_0) \tag{4.7}$$

For the case when only a hydrostatic pressure of magnitude p distributed over the surface acts from the external medium on the rigid core, then we obtain from (1.14) and (4.7)

$$-\boldsymbol{\omega} \cdot \boldsymbol{\Phi}_2 + \iint_{O'} \mathbf{R} \times \mathbf{f}\, dO = p\boldsymbol{\omega} \times \int_{\Gamma'} \mathbf{R}_0 (\mathbf{t} \cdot \mathbf{R}_0)\, dS \tag{4.8}$$
$$-\iint_{O'} \mathbf{f}\, dO = \frac{1}{2}\, p\boldsymbol{\omega} \times \int_{\Gamma'} \mathbf{t} \times \mathbf{R}_0 dS$$

The last formulas show that the conjugate conditions (4.6) are independent of the shape of the surface O' of the rigid inclusion in this case.

Conditions for the potentiality of the load (4.7) follow from the directly confirmable relationship

$$p \iint_O (\mathrm{tr}\,\varepsilon_0 \delta w - \boldsymbol{\vartheta} \cdot \delta\mathbf{w}_0)\, dO = \delta \left[\frac{1}{2}\, p \iint_O (w\, \mathrm{tr}\,\varepsilon_0 - \boldsymbol{\vartheta} \cdot \mathbf{u})\, dO \right] - \tag{4.9}$$
$$\frac{1}{2}\, p \int_{\Gamma} (\mathbf{t} \times \mathbf{w}_0) \cdot \delta\mathbf{w}_0\, dS$$

for the case of a uniform pressure distributed over the whole middle surface resting on the outline Γ. If the kinematic boundary conditions on Γ are such that the contour integral in (4.9) vanishes, then the elementary work of the load (4.7) is a total variation of the functional.

REFERENCES

1. Lur'e, A. I., Theory of Elasticity. "Nauka", Moscow, 1970.
2. Lur'e, A. I., Bifurcation of equilibrium of a perfectly elastic body. PMM Vol. 30, № 4, 1966.
3. Zubov, L. M., Variational principles of the nonlinear theory of elasticity. Case of superposition of a small deformation on a finite deformation. PMM Vol. 35, № 5, 1971.
4. Lur'e, A. I., On differentiation of tensors with respect to time. Trudy Leningrad Politechnic Institute. 1971.
5. Appel, P., Theoretical Mechanics, Vol. 1. Fizmatgiz, Moscow, 1960.
6. Kagan, V. F., Principles of the Theory of Surfaces in a Tensor Exposition, Vol. 1. Gostekhizdat, Moscow-Leningrad, 1947.
7. Novozhilov, V. V., Theory of Thin Shells. Sudpromgiz, Leningrad, 1951.
8. Timoshenko, S. P., Stability of Elastic Systems. Gostekhizdat, Moscow, 1955.

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ASYMPTOTIC ANALYSIS OF THE STATE OF STRESS OF AN INFINITE CIRCULAR CYLINDRICAL SHELL

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The transverse sections of a shell are deformed identically under the effect of external forces which do not vary along the generator. In this case it is admissible to limit oneself to a study of the state of stress of an elastic concentric ring. A large quantity of papers is devoted to this classical problem. Primarily the case of nonthin shells is treated. The exception is in the papers of Ustinov [1, 2], where the state of stress of a very thin ring subjected to normal forces is considered. The stress field in a thin ring, seemingly subjected to both normal and tangential external forces, is also analyzed in this paper by another method.

1. Let S designate a domain occupied by a concentric ring, and L_2 and L_1 its outer and inner bounding circles, respectively. We take the boundary conditions for the first fundamental problem in the usual form

$$\varphi_1(t) + \overline{t\varphi_1'(t)} + \overline{\psi_1(t)} = f_2(t) \quad \text{on} \quad L_2 \tag{1.1}$$

$$\varphi_1(t) + \overline{t\varphi_1'(t)} + \overline{\psi_1(t)} = f_1(t) + C_1 \quad \text{on} \quad L_1 \tag{1.2}$$

where $\varphi_1(z)$ and $\psi_1(z)$ are the required functions, regular in the domain S, and $f_1(t)$ and $f_2(t)$ are some functions given on the corresponding curves L_1 and L_2. Examination